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Optical Solitons in a Monomode Fiber

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We discuss the propagation of optical solitons in a monomode fiber as a model of long-distance–high-bit-rate transmission system. We give several new results which did not appear in our previous papers on this subject, such as (1) a derivation of the perturbed nonlinear Schrödinger equation from the Maxwell equation, (2) on the integrability of the perturbed nonlinear Schrödinger equation, (3) a discussion of the soliton as a stable fixed point of certain infinite-dimensional map generated by a transmission system with periodic excitations.

KEY WORDS: Solitons; nonlinear Schrödinger equation; perturbation method; reshaping.

1. INTRODUCTION

In an optical transmission system using linear pulses, the bit rate of transmission (i.e., channel capacity of an optical fiber) is limited by the dispersion character of the fiber material. The dispersion is the deterministic factor in deciding the rate of pulse broadening. To overcome this limitation, the nonlinear change of dielectric (the so-called Kerr effect) of the fiber has been used to compensate for the dispersion effect.[1] When the frequency shift due to the Kerr effect is balanced with that due to the dispersion, the optical pulse may tend to form a stable nonlinear pulse (called optical soliton). The optical solitons are now considered to have a potential application to a high-bit-rate transmission system as shown in Ref. 2. For the solitons, the fiber loss is the only factor that contributes to the deterioration of the pulse quality by broadening the pulse width.[2] In a series of papers[3–5] we have shown that the optical solitons deformed by

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the fiber loss can be reshaped to a narrower and higher pulse during the course of transmission through a fiber. The idea for the reshaping of solitons is based upon a unique property of solitons that the dispersion is balanced by the nonlinear effect. We proposed several models of long-distance transmission system, and showed numerically that the soliton can propagate free of distortions through a fiber with periodic excitation at appropriate distances.

In this paper, we show several new results (which did not appear in the previous papers[1-4]) concerning the optical solitons propagating through a monomode fiber. The paper consists of three parts:

First, starting from the Maxwell equation for the electric field in a fiber with an inhomogeneous dielectric constant, we derive the nonlinear Schrödinger (NLS) equation with higher-order terms (as the perturbation terms) in an appropriate asymptotic sense. Although there have been several publications dealing with this problem,[1,6] most of the papers appear to be inconsistent in several points, for example, (1) the TE or TM mode has been assumed for the electric or magnetic field (this is not valid when the dielectric constant is inhomogeneous), (2) the ordering of scales in the variables (field variable and coordinates) is not clear. The method used in this section is based on the asymptotic perturbation technique developed by Taniuti *et al.* (the so-called reductive perturbation method[7]), and gives a consistent scheme for the derivation of the NLS equation and the higher-order corrections. The main equation derived here is given by the following form:

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+|q|^2 q = \varepsilon i\left(\beta_1\frac{\partial^3 q}{\partial T^3}+\beta_2|q|^2\frac{\partial q}{\partial T}+\beta_3 q^2\frac{\partial q^*}{\partial T}\right)-i\Gamma q \tag{1.1}$$

which we call the perturbed nonlinear Schrödinger (PNLS) equation (see below for the meaning of the variables in the equation).

Second, we discuss several properties of the equation (1.1), and its solutions (especially, soliton), and how the perturbations on the right hand side of (1.1) affect the integrability of the NLS equation.

Finally, we consider a feasibility of a long-distance–high-bit-rate optical transmission system by use of solitons. In order to establish such a system, we discuss a possibility that the soliton can be considered as a stable fixed point of an infinite-dimensional map corresponding to a transmission system having periodic excitations.

2. DERIVATION OF THE PNLS EQUATION (1.1)

In this section, by introducing an appropriate scale of coordinates based on the physical setting, we reduce the Maxwell equation (three-dimensional vector equations) into the PNLS equation (one-dimensional scalar equation) describing the optical pulse propagating through a cylindrical fiber.

The electric field $\mathbf{E}$ in an optical fiber with the dielectric constant ϵ satisfies the Maxwell equation,

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{D} \tag{2.1}$$

where c is the speed of light, and the displacement $\mathbf{D} = \epsilon * \mathbf{E}$ may be given in the following form:

$$\begin{aligned}(\epsilon * \mathbf{E})(t) = &\int_{-\infty}^{t} dt_1\, \epsilon^{(0)}(t-t_1)\, \mathbf{E}(t_1) \\ &+ \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3\, \epsilon^{(2)}(t-t_1, t-t_2, t-t_3) \\ &\times [\mathbf{E}(t_1)\cdot \mathbf{E}(t_2)]\, \mathbf{E}(t_3) \\ &+ \text{higher nonlinear terms}\end{aligned} \tag{2.2}$$

Here the second term indicates the Kerr effect, and the coefficients $\epsilon^{(0)}$, $\epsilon^{(2)}$ depend also on the spatial coordinates implying the inhomogenity in dielectric constant. By virtue of the formula of the vector calculus, Eq. (2.1) can be written in the form

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{D} = \nabla(\nabla \cdot \mathbf{E}) \tag{2.3}$$

It should be noted that $\nabla \cdot \mathbf{E}$ in (2.3) is not zero, since $\nabla \cdot \mathbf{D} = 0$ (constraint for $\mathbf{D}$ in Maxwell's equation) implies $\epsilon * (\nabla \cdot \mathbf{E}) = -(\nabla\epsilon *)\cdot \mathbf{E} \neq 0$. Namely, the electric field is *not* TE mode, and (2.3) cannot be reduced simply to a scalar equation (e.g., by assuming $\mathbf{E} = \nabla\Psi \times \mathbf{e}$, $\mathbf{e}$ a unit vector in the direction of propagation). However as far as the linear problem is concerned, $\nabla \cdot \mathbf{E}$ can be ignored, since in the practical case the order of $\nabla\epsilon(\sim \nabla \cdot \mathbf{E}/|\mathbf{E}|)$ is small, $O(\nabla\epsilon) \simeq 10^{-3}$ for monomode fibers.[8] [Note that the order of nonlinearity considered here is much smaller than $O(\nabla\epsilon)$, that is, the right-hand side of (2.3) cannot be ignored when the nonlinear problem is considered.]

For our purpose to reduce (2.3) in the sense of asymptotic perturbation method, it is convenient to write (2.3) in the following matrix form:

$$L\mathsf{E}=0 \tag{2.4}$$

where E expresses a column vector, i.e., $\mathsf{E}={}^t(E_x, E_y, E_z)$, and in the cylindrical coordinates with the z axis as the axial direction of the fiber, the matrix L consisting of the three parts $L=L_a+L_b-L_c$ is defined by

$$L_a=\begin{pmatrix} \nabla_\perp^2-\frac{1}{r^2} & -\frac{2}{r^2}\frac{\partial}{\partial\theta} & 0 \\ \frac{2}{r^2}\frac{\partial}{\partial\theta} & \nabla_\perp^2-\frac{1}{r^2} & 0 \\ 0 & 0 & \nabla_\perp^2 \end{pmatrix} \tag{2.5a}$$

$$L_b=\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\epsilon *\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5b}$$

$$L_c=\begin{pmatrix} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r & \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta} & \frac{\partial^2}{\partial r\partial z} \\ \frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}r & \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial z}r & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix} \tag{2.5c}$$

Note that these matrices imply

$$L_a\mathsf{E}=\nabla_\perp^2\,\mathbf{E}=\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\mathbf{E}$$

$$L_b\mathsf{E}=\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\epsilon *\right)\mathbf{E}$$

and

$$L_c\mathsf{E}=\nabla(\nabla\cdot\mathbf{E})$$

We consider the electric field as a nearly monochromatic wave propagating along the z axes with the wave number k_1 and angular frequency ω_1, that is, the field E is assumed to be in the expansion form,

$$\mathsf{E}(r,\theta,z,t)=\sum_{l=-\infty}^{\infty}\mathsf{E}_l(r,\theta,\xi,\tau;\varepsilon)\exp[i(k_l z-\omega_l t)] \tag{2.6}$$

with $\mathsf{E}_{-l} = \mathsf{E}_l^*$ (complex conjugate), where $k_l = lk_1$, $\omega_l = l\omega_1$ and the summation is taken over all harmonics generated by the nonlinearity due to the Kerr effect, and $\mathsf{E}_l(r, \theta, \xi, \tau; \varepsilon)$ is the envelope of lth harmonic changing slowly in z and t. Here the slow variables ξ and τ are defined by

$$\xi = \varepsilon^2 z, \qquad \tau = \varepsilon\left(t - \frac{z}{v_g}\right) \tag{2.7}$$

where the small parameter $\varepsilon(|\varepsilon| \ll 1)$ expresses the order of the nonlinearity (i.e., the order of the electric field) and v_g is the group velocity of the wave given below. Since the radius of the fiber has the same order as the wavelength $(2\pi/k_1)$, the scale for the transverse coordinates (r, θ) is of order 1. In this scale of the coordinates (2.7), we are looking at a behavior of the field in the balance between the nonlinearity and the dispersion which results in the forming of optical solitons confined in the transverse direction. With (2.6) and (2.7), the displacement $\mathsf{D} = \epsilon * \mathsf{E} = \sum \mathsf{D}_l \exp[i(k_l z - \omega_l t)]$ is given by

$$\begin{aligned}
\mathsf{D}_l(r, \theta, \xi, \tau; \varepsilon) = {} & \epsilon_l^{(0)} \mathsf{E}_l + \varepsilon i \dot{\epsilon}_l^{(0)} \frac{\partial \mathsf{E}_l}{\partial \tau} - \varepsilon^2 \frac{1}{2} \ddot{\epsilon}_l^{(0)} \frac{\partial^2 \mathsf{E}_l}{\partial \tau^2} \\
& - \varepsilon^3 \frac{i}{3!} \dddot{\epsilon}_l^{(0)} \frac{\partial^3 \mathsf{E}_l}{\partial \tau^3} + [\epsilon_{l_1 l_2 l_3}^{(2)} (\mathsf{E}_{l_1} \cdot \mathsf{E}_{l_2})\, \mathsf{E}_{l_3}]_{l_1 + l_2 + l_3 = l} \\
& + i \Bigg[\dot{\epsilon}_{l_1 l_2 l_3}^{(2)} \left(\frac{\partial \mathsf{E}_{l_1}}{\partial \tau} \cdot \mathsf{E}_{l_2} \right) \mathsf{E}_{l_3} + \dot{\epsilon}_{l_1 l_2 l_3}^{(2)} \left(\mathsf{E}_{l_1} \cdot \frac{\partial \mathsf{E}_{l_2}}{\partial \tau} \right) \mathsf{E}_{l_3} \\
& + \dot{\epsilon}_{l_1 l_2 l_3}^{(2)} (\mathsf{E}_{l_1} \cdot \mathsf{E}_{l_2}) \frac{\partial \mathsf{E}_{l_3}}{\partial \tau} \Bigg]_{l_1 + l_2 + l_3 = l} + \cdots
\end{aligned} \tag{2.8}$$

where $\epsilon_l^{(0)}$ is the Fourier coefficient $\tilde{\epsilon}^{(0)}(\Omega)$ of $\epsilon^{(0)}(t)$ at $\Omega = \omega_l$, i.e., $\epsilon_l^{(0)} = \tilde{\epsilon}^{(0)}(\omega_l)$, and $\dot{\epsilon}_l^{(0)} = \partial \epsilon_l^{(0)} / \partial \omega_l, \ldots,$ and $\epsilon_{l_1 l_2 l_3}^{(2)}$ is the Fourier coefficient $\tilde{\epsilon}^{(2)}(\Omega_1, \Omega_2, \Omega_3)$ of $\epsilon^{(2)}(t_1, t_2, t_3)$ at $\Omega_1 = \omega_{l_1}$, $\Omega_2 = \omega_{l_2}$, $\Omega_3 = \omega_{l_3}$, and $\dot{\epsilon}_{l_1 l_2 l_3}^{(2)} = \partial \epsilon_{l_1 l_2 l_3}^{(2)} / \partial \omega_{l_1}$ and so on.

We now assume that $\mathsf{E}_l(r, \theta, \xi, \tau; \varepsilon)$ can be expanded in terms of ε,

$$\mathsf{E}_l(r, \theta, \xi, \tau; \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \mathsf{E}_l^{(n)}(r, \theta, \xi, \tau) \tag{2.9}$$

Then, from (2.4), (2.6), (2.7), and (2.8) we have, at order ε,

$$L_l \mathsf{E}_l^{(1)} = 0 \tag{2.10}$$

where L_l is L with the replacements $\partial/\partial z = ik_l$, $\partial/\partial t = -i\omega_l$, $\epsilon * = \epsilon_l^{(0)}$. Note that the operator L_l is self-adjoint, $L_l^+ = L_l$, in the sense of the following inner product:

$$(\mathsf{U}, \mathsf{V}) = \int_D \mathsf{U}^+ \mathsf{V}\, dS \tag{2.11}$$

where $dS = r dr d\theta$, D is the cross section of the fiber, and $A^+ =$ the adjoint of $A = (A_{ij})^+ = (A_{ji}^*)$. In the equation (2.10), we consider the case (of monomode fiber) in which there is only one bound state with the eigenvalue k_1^2 (i.e., $l = \pm 1$) and the eigenfunction $\mathsf{U} = \mathsf{U}(r, \theta)$ (called the mode function describing the confinement of the pulse in the transverse direction), that is, the solution to (2.10) can be written in

$$\mathsf{E}_l^{(1)}(r, \theta, \xi, \tau) = \begin{cases} q_1^{(1)}(\xi, \tau)\, \mathsf{U}(r, \theta), & \text{for } l = 1 \\ 0, & \text{for } l \neq \pm 1 \end{cases} \tag{2.12}$$

Here the coefficient $q_1^{(1)}(\xi, \tau)$ with $q_{-1}^{(1)} = q_1^{(1)*}$ is a complex scalar function satisfying certain equations given in the higher-order equation of (2.4). From the equation $L_1 \mathsf{U} = 0$, the inner product $(\mathsf{U}, L_1 \mathsf{U}) = 0$ gives the linear dispersion relation $k_1 = k_1(\omega_1)$,

$$k_1^2 = \frac{\omega_1^2}{c^2} (\mathsf{U}, n_0^2 \mathsf{U}) + (\mathsf{U}, L_0 \mathsf{U}) \tag{2.13}$$

where $n_0 = (\epsilon_1^{(0)})^{1/2}$ is the index of refraction, and we have assumed the normalization for U by $U_x^2 + U_y^2 = 1$.

At order ε^2, we have

$$L_l \mathsf{E}_l^{(2)} = i \left[-\frac{\partial L_l}{\partial \omega_l} + \left(\frac{1}{v_g} - \frac{\partial k_1}{\partial \omega_1} \right) \frac{\partial}{\partial \omega_l} \left(L_l - \frac{\omega_l^2}{c^2} \epsilon_l^{(0)} \right) \right] \frac{\partial \mathsf{E}_l^{(1)}}{\partial \tau} \tag{2.14}$$

from which we obtain $\mathsf{E}_l^{(2)} = 0$ if $l \neq \pm 1$. In the case $l = 1$, it is required that the inhomogeneous equation (2.14) satisfies the compatibility (or integrability) condition,

$$(\mathsf{U}, L_1 \mathsf{E}_1^{(2)}) = 0 \tag{2.15}$$

This gives the group velocity v_g in terms of the linear dispersion relation (2.13),

$$\frac{1}{v_g} = \frac{\partial k_1}{\partial \omega_1} \tag{2.16}$$

and (2.14) for $l=1$ becomes

$$L_1 \mathsf{E}_1^{(2)} = -i \frac{\partial L_1}{\partial \omega_1} \frac{\partial \mathsf{E}_1^{(1)}}{\partial \tau} = -i \frac{\partial L_1}{\partial \omega_1} \frac{\partial q_1^{(1)}}{\partial \tau} \mathsf{U} \tag{2.17}$$

From (2.10) for $l=1$, one can find the solution of (2.17) in the form

$$\mathsf{E}_1^{(2)} = i \frac{\partial q_1^{(1)}}{\partial \tau} \frac{\partial \mathsf{U}}{\partial \omega_1} + q_1^{(2)} \mathsf{U} \tag{2.18}$$

where $q_1^{(2)} = q_1^{(2)}(\xi, \tau)$ with $q_{-1}^{(2)} = q_1^{(2)*}$ is a scalar function to be determined in the higher-order equation.

At order ε^3, we have

$$L_l \mathsf{E}_l^{(3)} \begin{cases} = 0, & \text{if } \; l \neq \pm 1, \pm 3 \\ = -\dfrac{9\omega_1^2}{c^2} \epsilon_3^{(2)} q_1^{(1)3} (\mathsf{U} \cdot \mathsf{U})\, \mathsf{U}, & \text{if } \; l = 3 \\ = -i \dfrac{\partial L_1}{\partial \omega_1} \dfrac{\partial \mathsf{E}_1^{(2)}}{\partial \tau} + \dfrac{1}{2} \dfrac{\partial^2 L_1}{\partial \omega_1^2} \dfrac{\partial^2 \mathsf{E}_1^{(1)}}{\partial \tau^2} & \\ \quad + \left(i \dfrac{\partial q_1^{(1)}}{\partial \tau} - \dfrac{1}{2} \dfrac{\partial^2 k_1}{\partial \omega_1^2} \dfrac{\partial^2 q_1^{(1)}}{\partial \tau^2} \right) & \\ \quad \times \left[\dfrac{\partial}{\partial k_1} \left(L_1 - \dfrac{\omega_1^2 n_0^2}{c^2} \right) \right] \mathsf{U} & \\ \quad - |q_1^{(1)}|^2 q_1^{(1)} \dfrac{\omega_1^2 \varepsilon_1^{(2)}}{c^2} (\mathsf{U} \cdot \mathsf{U})\, \mathsf{U}, & \text{if } \; l = 1 \end{cases} \tag{2.19}$$

where $\epsilon_l^{(2)} = \sum_{l_1 + l_2 + l_3 = l} \epsilon_{l_1 l_2 l_3}^{(2)} (l_i = \pm 1)$. (Note that $\epsilon_1^{(2)}$ is a positive real number for the Kerr effect.) From (2.19), one can obtain the solutions, $\mathsf{E}_l^{(3)} = 0$ for $l \neq \pm 1$ or ± 3, and since L_3 does not have the eigenmode (i.e., ker $L_3 = 0$),

$$\mathsf{E}_3^{(3)} = -\frac{9\omega_1^2}{c^2} q_1^{(1)3} L_3^{-1} [\epsilon_3^{(2)} (\mathsf{U} \cdot \mathsf{U})\, \mathsf{U}] \tag{2.20}$$

which is a harmonic generated by the nonlinearity. For $l=1$, we again require the compatibility condition

$$(\mathsf{U}, L_1 \mathsf{E}_1^{(3)}) = 0 \tag{2.21}$$

from which we obtain the NLS equation for $q_1^{(1)}(\xi, \tau)$,

$$i\frac{\partial q_1^{(1)}}{\partial \xi}-\frac{1}{2}\frac{\partial^2 k_1}{\partial \omega_1^2}\frac{\partial^2 q_1^{(1)}}{\partial \tau^2}+\nu\,|q_1^{(1)}|^2\,q_1^{(1)}=0 \tag{2.22}$$

where ν is a positive real number given by

$$\nu=\frac{\omega_1^2}{2k_1 c^2}(\mathsf{U}, \epsilon_2^{(2)}\{\mathsf{U}\cdot\mathsf{U}\}\,\mathsf{U}) \tag{2.23}$$

Hence, the optical soliton can propagate through the fiber when $\ddot{k}_1=\partial^2 k_1/\partial\omega_1^2<0$ (i.e., the abnormal dispersion)[1]. Here, it is worth noting that the explicit form (2.18) for $\mathsf{E}_1^{(2)}$ does not need in the calculation of the compatibility condition (2.21), and that (2.21) can be calculated directly from the equations for $\mathsf{E}_1^{(1)}$ and $\mathsf{E}_1^{(2)}$, i.e., (2.10) and (2.17).

In order to see the effects of the higher-order terms, one needs to find the equation for $q_1^{(2)}$ in (2.18). For this purpose, we have, at order ε^4, the equation $L_l\,\mathsf{E}_l^{(4)}$ for $l=1$,

$$\begin{aligned}
L_1\,\mathsf{E}_1^{(4)}=&-i\frac{\partial L_1}{\partial\omega_1}\frac{\partial \mathsf{E}_1^{(3)}}{\partial\tau}+\frac{1}{2}\frac{\partial^2 L_1}{\partial\omega_1^2}\frac{\partial^2\mathsf{E}_1^{(2)}}{\partial\tau^2}+\frac{i}{6}\frac{\partial^3 L_1}{\partial\omega_1^3}\frac{\partial^3\mathsf{E}_1^{(1)}}{\partial\tau^3}\\
&+\left[\frac{\partial}{\partial k_1}\left(L_1-\frac{\omega_1^2 n_0^2}{c^2}\right)\right]\\
&\times\left(i\frac{\partial\mathsf{E}_1^{(2)}}{\partial\xi}-\frac{1}{2}\frac{\partial^2 k_1}{\partial\omega_1^2}\frac{\partial^2\mathsf{E}_1^{(2)}}{\partial\tau^2}-\frac{i}{6}\frac{\partial^3 k_1}{\partial\omega_1^3}\frac{\partial^3\mathsf{E}_1^{(1)}}{\partial\tau^3}\right)\\
&+i\left[\frac{\partial}{\partial\omega_1}\left(L_1-\frac{\omega_1^2 n_0^2}{c^2}\right)\right]\frac{\partial}{\partial\tau}\left(i\frac{\partial\mathsf{E}_1^{(1)}}{\partial\xi}-\frac{i}{2}\frac{\partial^2 k}{\partial\omega_1^2}\frac{\partial^2\mathsf{E}_1^{(1)}}{\partial\tau^2}\right)\\
&-\frac{\omega^2}{c^2}\Bigg[\epsilon_{l_1l_2l_3}^{(2)}\sum_{i+j+k=4}(\mathsf{E}_{l_1}^{(i)}\cdot\mathsf{E}_{l_2}^{(j)})\,\mathsf{E}_{l_3}^{(k)}\\
&+i\sum_{i=1}^{3}\frac{\partial\epsilon_{l_1l_2l_3}^{(2)}}{\partial\omega_{l_i}}\frac{\partial}{\partial\tau_i}(\mathsf{E}_{l_i}^{(1)}\cdot\mathsf{E}_{l_2}^{(1)})\,\mathsf{E}_{l_3}^{(1)}\Bigg]_{l_1+l_2+l_3=1}
\end{aligned} \tag{2.24}$$

where $\partial[(\mathsf{E}_{l_1}^{(1)}\cdot\mathsf{E}_{l_2}^{(1)})\,\mathsf{E}_{l_3}^{(1)}]/\partial\tau_1=(\partial\mathsf{E}_{l_1}^{(1)}/\partial\tau\cdot\mathsf{E}_{l_2}^{(1)})\,\mathsf{E}_{l_3}^{(1)}$ and so on. Using (2.10), (2.17), (2.19) and the remark below (2.23) [i.e., without solving (2.19) for $\mathsf{E}_1^{(3)}$], one can calculate the compatibility condition for (2.24), i.e., $(\mathsf{U}, L_1\,\mathsf{E}_1^{(4)})=0$. The resulting equation for $q_1^{(2)}$ from the compatibility condition is

$$\begin{aligned}
&i\frac{\partial q_1^{(2)}}{\partial\xi}-\frac{1}{2}\frac{\partial^2 k_1}{\partial\omega_1^2}\frac{\partial^2 q_1^{(2)}}{\partial\tau^2}+2\nu\,|q_1^{(1)}|^2\,q_1^{(2)}+\nu q_1^{(1)2}q_1^{(2)*}\\
&\quad=i\alpha_1\frac{\partial^3 q_1^{(1)}}{\partial\tau^3}+i\alpha_2\,|q_1^{(1)}|^2\frac{\partial q_1^{(1)}}{\partial\tau}+i\alpha_3 q_1^{(1)}\frac{\partial q_1^{(1)*}}{\partial\tau}
\end{aligned} \tag{2.25}$$

where $\alpha_i (i = 1, 2, 3)$ are real constants given by

$$\alpha_1 = \frac{1}{6} \frac{\partial^3 k_1}{\partial \omega_1^3} \tag{2.26a}$$

$$\alpha_2 = -\frac{\omega_1^2}{c^2} \left(\mathrm{U}, \left\{ \epsilon_1^{(2)} \frac{\partial}{\partial \omega_1} \left[\frac{1}{k_1} (\mathrm{U} \cdot \mathrm{U}) \right] + \epsilon_1^{(2)} (\mathrm{U} \cdot \mathrm{U}) \right\} \mathrm{U} \right) \tag{2.26b}$$

$$\alpha_3 = \frac{\omega_1^2}{2k_1^2 c^2} \left(\mathrm{U}, \left\{ \epsilon_1^{(2)} \frac{\partial}{\partial \omega_1} [k_1 (\mathrm{U} \cdot \mathrm{U})] - \epsilon_{-1}^{(2)} (\mathrm{U} \cdot \mathrm{U}) \right\} \mathrm{U} \right) \tag{2.26c}$$

with $\epsilon_1^{(2)} \equiv \epsilon_{11-1}^{(2)} + \epsilon_{-111}^{(2)} + \epsilon_{1-11}^{(2)}$ and $\epsilon_{-1}^{(2)} \equiv \epsilon_{-111}^{(2)} + \epsilon_{1-11}^{(2)} + \epsilon_{11-1}^{(2)}$. Through the paper, we study an equation for a function $q(T, Z)$ in the form (1.1), which is equivalent to the equation combined (2.22) and (2.25) with the normalizations, $T = \tau(-2\pi \ddot{k}_1/k_1)^{-1/2}$, $z = \xi(k_1/2\pi)$, $q = \sqrt{v}(q_1^{(1)} + \varepsilon q_1^{(2)})$, $\beta_1 = \alpha_1(2\pi/k_1)(-2\pi \ddot{k}_1/k_1)^{-3/2}$, $\beta_2 = \alpha_2(2\pi/vk_1)(-2\ddot{k}_1/k_1)^{-1/2}$, $\beta_3 = \alpha_3(2\pi/vk_1)(-2\pi \ddot{k}_1/k_1)^{-1/2}$ and $\Gamma = 0$. The fiber loss is given by an imaginary part of the dielectric constant $\epsilon_1^{(0)}$, and may appear as the term in (1.1).[2] In the practical cases, the order of Γ can be assumed to be ε, $\Gamma = \varepsilon \Gamma_0$ and $O(\Gamma_0) \sim 1$.

3. EFFECTS OF PERTURBATIONS

In this section, we first briefly summarize the facts of the NLS equation as a completely integrable system, and then study the effects of perturbations based on the PNLS equation derived in the previous section.

The NLS equation considered here is given by the normalized form (1.1) with $\varepsilon = \Gamma = 0$, i.e.,

$$i \frac{\partial q}{\partial Z} + \frac{1}{2} \frac{\partial^2 q}{\partial T^2} + |q|^2 q = 0 \tag{3.1}$$

For a certain class of functions $q_0(T) = q(T, 0)$, the initial conditions, [e.g., $|q_0(T)|$ approaches to zero sufficiently rapidly as $|T|$ goes to infinity], one can solve the initial value problem of (3.1) by means of the method of inverse scattering transform (IST). The general response $q(T, Z)$ calculated by the IST method consists of N-number of solitons (N-soliton solution) and radiations (linearlike dispersive wave) which die off asymptotically as $Z \to \infty$.[9] The one-soliton solution in the general form is given by

$$q(T, Z) = \eta \operatorname{sech} \eta(T + \kappa Z - \theta_0) \exp\left[-i\kappa T + \frac{i}{2}(\eta^2 - \kappa^2) Z - i\sigma_0 \right] \tag{3.2}$$

where η, κ, θ_0, and σ_0 are arbitrary constants determined from the initial condition.[9] As mentioned before, such a solution appears due to the balance of the dispersion effect and the nonlinear response due to the Kerr effect. This balance can be also seen in (3.2) as the relation between the amplitude η and the pulsewidth $\sim 1/\eta$. One of the important properties of the soliton for the communication systems is its stability. The soliton is known to be a stable pulse against the noise and some perturbations.[1] This is a consequence of the fact that the NLS equation has an infinite number of conserved quantities. In this sense, we often say that the NSL equation is completely integrable. The several conserved quantities are given by

$$\begin{aligned} \mathscr{H}_0 &= \int_{-\infty}^{\infty} |q|^2 \, dT && \text{(energy)} \\ \mathscr{H}_1 &= i \int_{-\infty}^{\infty} q \frac{\partial q^*}{\partial T} \, dT && \text{(energy flux)} \\ \mathscr{H}_1 &= \int_{-\infty}^{\infty} \left(\left| \frac{\partial q}{\partial T} \right|^2 - |q|^4 \right) dT && \text{(Hamiltonian)} \end{aligned} \tag{3.3}$$

In terms of the one-soliton solution, these quantities are $\mathscr{H}_0 = 2\eta$, $\mathscr{H}_1 = -2\kappa\eta$, $\mathscr{H}_2 = 2\kappa^2\eta - 2\eta^3/3$.

We now discuss a solution of the PNLS equation (1.1). Here we employ the perturbation method developed in Ref. 10 to study the behavior of the soliton under the influence of the perturbations in (1.1). At the end of this section, we briefly note a recent result on the integrability of the PNLS equation without the loss term (i.e., $\Gamma = 0$).

The basic idea of the perturbation method in Ref. 10 is to assume the solution of (1.1) to be the following form (called a quasistationary solution):

$$q(T, Z) = \hat{q}(\theta, Z_1; \varepsilon) \exp[-i\kappa(\theta - \theta_0) + i(\sigma - \sigma_0)] \tag{3.4}$$

where Z_1 is a slow variable defined by $Z_1 = \varepsilon Z$, and θ, σ are the fast variables satisfying

$$\begin{aligned} &\frac{\partial \theta}{\partial T} = 1, \qquad \frac{\partial \theta}{\partial Z} = \kappa \\ &\frac{\partial \sigma}{\partial T} = 0, \qquad \frac{\partial \sigma}{\partial Z} = \frac{1}{2}(\eta^2 + \kappa^2) \end{aligned} \tag{3.5}$$

Because of the perturbations, the four parameters of the soliton (3.2) $\{\eta, \kappa, \theta_0, \sigma_0\}$ become the functions of Z_1. Namely, assuming the quasistationarity (3.4), we expect that the solution of the PNLS equation evolves adiabatically. Substituting (3.4) into (1.1), we have an equation for $\hat{q}$,

$$\frac{1}{2}\frac{\partial^2 \hat{q}}{\partial \theta^2} + |\hat{q}|^2 \hat{q} - \frac{1}{2}\eta^2 \hat{q} = \varepsilon F(\hat{q}) \tag{3.6}$$

where the right-hand side is given by

$$\begin{aligned} F(\hat{q}) = & -i\frac{\partial \hat{q}}{\partial Z_1} - \left\{\frac{\partial}{\partial Z_1}\left[(\theta-\theta_0)\kappa + \sigma_0\right] + \beta_1\kappa^3 - \beta_2\kappa|\hat{q}|^2 + i\Gamma_0\right\}\hat{q} \\ & - \beta_3\kappa\hat{q}^2\hat{q}^* - i(3\beta_1\kappa^2 - \beta_2|\hat{q}|^2)\frac{\partial \hat{q}}{\partial \theta} + i\beta_3\hat{q}^2\frac{\partial \hat{q}^*}{\partial \theta} \\ & + 3\beta_1\kappa\frac{\partial^2 \hat{q}}{\partial \theta^2} + i\beta_1\frac{\partial^3 \hat{q}}{\partial \theta^3} \end{aligned} \tag{3.7}$$

Let us assume that $\hat{q}$ can be expanded in terms of ε,

$$\hat{q}(\theta, Z_1; \varepsilon) = \hat{q}^{(0)}(\theta, Z_1) + \varepsilon\hat{q}^{(1)}(\theta, Z_1) + \cdots \tag{3.8}$$

where the first term $\hat{q}^{(0)}(\theta, Z_1)$ is given by the soliton form (3.2),

$$\hat{q}^{(0)}(\theta, Z_1) = \eta \operatorname{sech} \eta(\theta - \theta_0) \tag{3.9}$$

From (3.6) and (3.8), and at order ε, setting $\hat{q}^{(1)} = \varphi + i\psi$ where φ and ψ are real functions, we have

$$\begin{aligned} L_R\varphi &= \left(\frac{1}{2}\frac{\partial^2}{\partial \theta^2} + 3\hat{q}^{(0)2} - \frac{1}{2}\eta^2\right)\varphi = \operatorname{Re}F(\hat{q}^{(0)}) \\ L_I\psi &= \left(\frac{1}{2}\frac{\partial^2}{\partial \theta^2} + \hat{q}^{(0)2} - \frac{1}{2}\eta^2\right)\psi = \operatorname{Im}F(\hat{q}^{(0)}) \end{aligned} \tag{3.10}$$

where $\operatorname{Re}F(\hat{q}^{(0)})$ and $\operatorname{Im}F(\hat{q}^{(0)})$ are the real and the imaginary parts of $F(\hat{q}^{(0)})$. Noticing that the operators L_R and L_I in (3.10) are self-adjoint, and $L_R(\partial\hat{q}^{(0)}/\partial\theta) = 0$, $L_I\hat{q}^{(0)} = 0$, the compatibility conditions for (3.10) are given by

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\partial \hat{q}^{(0)}}{\partial \theta} \operatorname{Re}F(\hat{q}^{(0)})\, d\theta = 0 \\ &\int_{-\infty}^{\infty} \hat{q}^{(0)} \operatorname{Im}F(\hat{q}^{(0)})\, d\theta = 0 \end{aligned} \tag{3.11}$$

The conditions (3.11) lead to the equations for η and κ,

$$\frac{d\eta}{dZ_1} = -2\Gamma_0\eta, \qquad \frac{d\kappa}{dZ_1} = 0 \tag{3.12}$$

Thus the amplitude of soliton decays as $\exp(-2\Gamma Z)$, but the velocity remains constant. Using these results, one can find the particular solutions of (3.10),

$$\begin{aligned}
\varphi_p &= \frac{1}{\eta}\left[\frac{d}{dZ_1}(\kappa\theta_0 - \sigma_0) + \beta_1\kappa(3\eta^2 - \kappa^2)\right]\frac{\partial \hat{q}^{(0)}}{\partial \eta} \\
&\quad + \kappa\left[-3\beta_1 + \frac{1}{2}(\beta_2 - \beta_3)\right]\hat{q}^{(0)} \\
\psi_p &= \left[\frac{d\theta_0}{dZ_1} + \beta_1(\eta^2 - 3\kappa^2)\right](\theta - \theta_0)\,\hat{q}^{(0)} \\
&\quad + \left[3\beta_1 - \frac{1}{2}(\beta_2 + \beta_3)\right]\frac{\partial \hat{q}^{(0)}}{\partial \theta} + \Gamma_0(\theta - \theta_0)^2\,\hat{q}^{(0)}
\end{aligned} \tag{3.13}$$

Note that (3.13) are valid for the region $|\theta - \theta_0| < O(\Gamma^{-1/2})$ only, because of the nonuniformity due to the fiber loss Γ. Whereas the terms $\partial \hat{q}^{(0)}/\partial \eta$ and $(\theta - \theta_0)\,\hat{q}^{(0)}$ do *not* cause the perturbation scheme to be nonuniform (even these look so), since these terms can be absorbed into the leading order solution $\hat{q}^{(0)}$ by shifting η and κ, respectively. To determine the parameters θ_0 and σ_0, one needs to consider the initial value problem for (1.1). The PNLS equation (1.1) derives the following equations for the conserved quantities of the NLS equation, (3.3):

$$\begin{aligned}
\frac{d}{dZ}\mathscr{H}_0 &= -2\Gamma\mathscr{H}_0 \\
\frac{d}{dZ}\hat{\mathscr{H}}_1 &= -2\Gamma\mathscr{H}_1 - 6\varepsilon^2\beta_1\beta_3\int_{-\infty}^{\infty}\left|\frac{\partial q}{\partial T}\right|^2\frac{\partial}{\partial T}|q|^2\,dT
\end{aligned} \tag{3.14}$$

where

$$\hat{\mathscr{H}}_1 = \mathscr{H}_1 - \varepsilon\beta_3\int_{-\infty}^{\infty}|q|^4\,dT \tag{3.15}$$

Using (3.8) and (3.12), at order ε, the equations (3.14) with the initial pulse as the soliton solution (3.2) give the conditions

$$\begin{aligned}
&\int_{-\infty}^{\infty}\varphi\hat{q}^{(0)}\,d\theta = 0 \\
&\int_{-\infty}^{\infty}\psi\frac{\partial \hat{q}^{(0)}}{\partial \theta}\,d\theta = \beta_3(\eta_0^2 - \eta^2)\,\eta
\end{aligned} \tag{3.16}$$

where $\eta_0 = \eta(0)$. From (3.13), we have the equations for θ_0 and σ_0,

$$\begin{aligned} \frac{d\theta_0}{dZ_1} &= (\eta^2 + 3\kappa^2)\,\beta_1 - \frac{1}{3}\eta^2\beta_2 + \left(\frac{2}{3}\eta^2 - \eta_0^2\right)\beta_3 \\ \frac{d\sigma_0}{dZ_1} &= 2(\kappa^2 - \eta^2)\,\kappa\beta_1 - \frac{2}{3}\kappa\eta^2\beta_2 - \left(\frac{1}{3}\eta^2 + \eta_0^2\right)\kappa\beta_3 \end{aligned} \tag{3.17}$$

For a simple case as an important example, the solution with the initial conditions $\eta = \eta_0$, $\kappa = \theta_0 = \sigma_0 = 0$ in (3.2) is given by

$$\begin{aligned} q(T, Z) &= q^{(0)}(T, Z) + \varepsilon q^{(1)}(T, Z) + O(\varepsilon^2) \\ &= \eta \operatorname{sech} \eta\hat{\theta} e^{i\hat{\sigma}} \\ &\quad + \varepsilon i\eta \left[\Gamma_0 \hat{\theta}^2 - \left(3\beta_1 - \frac{1}{2}\beta_2 - \frac{1}{2}\beta_3\right) \eta \tanh \eta\hat{\theta} \right] \operatorname{sech} \eta\hat{\theta} e^{i\hat{\sigma}} \\ &\quad + O(\varepsilon^2), \qquad \text{for} \quad |0| < O(\Gamma^{-1/2}) \quad \text{and} \quad 0 < Z < O(\varepsilon^{-1}) \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \hat{\theta} &= T - \frac{\eta_0^2}{4\Gamma_0}\left(\beta_1 - \frac{1}{3}\beta_2 + \frac{2}{3}\beta_3\right)(1 - e^{-4\Gamma Z}) - \varepsilon\eta_0^2 Z \\ \hat{\sigma} &= \frac{\eta_0^2}{8\Gamma}(1 - e^{-4\Gamma Z}) + \left[\left(2\beta_1 - \frac{1}{3}\beta_2 + \frac{2}{3}\beta_3\right)\eta^2 - \eta_0^2\right]\hat{\theta} \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), one can see that the higher-order dispersion and nonlinearity characterized $\beta_i (i = 1, 2, 3)$ modify the velocity, and deform the shape of soliton, but only by order ε for all $Z \leqslant (\varepsilon^{-1})$. It should be noted that the velocity modified by these perturbations depends on the soliton parameter η, and therefore the bound state soliton (where each of the solitons has the different η but the same κ) decays into a series of moving solitons. The main destruction of the soliton shape is caused by the fiber loss, which gives the limitation of the bit rate of the soliton transmission (see Ref. 2 for details). In the next section, based upon a unique property of solitons that the dispersion which depends on the pulse width is balanced by the nonlinearity, we discuss the reshaping of the soliton deformed by the loss.

Before ending this section, we note some remarks on the case of no fiber loss, $\Gamma = 0$. In this case, as mentioned above, the soliton is stable in the sense that the distortion of the shape remains order ε even after the distance of propagation $Z \sim O(\varepsilon^{-1})$ (unlike the linear dispersive pulse). It is also important and interesting to notice that the solution (3.18) with $\Gamma = 0$

gives a uniform solution in T of the PNLS equation (1.1), i.e., the solution is valid for all T, and the equation $q^{(0)}$ [soliton part in (3.18)] satisfies the following equation (instead of the NLS equation):

$$i\frac{\partial Q}{\partial Z}+\frac{1}{2}\frac{\partial^3 Q}{\partial T^2}+|Q|^2 Q=\varepsilon i\beta_1\left(\frac{\partial^3 Q}{\partial T^3}+6|Q|^2\frac{\partial Q}{\partial T}\right)+O(\varepsilon^2) \qquad (3.20)$$

The equation (3.20) is sometimes called the higher-order (or the hierarchy of) nonlinear Schrödinger (HNLS) equation which is also completely integrable by means of the IST method with the *same* eigenvalue problem for the NLS equation, and its conserved quantities are the same as those of the NLS equation. The general form of one-soliton solution for (3.20) is given by

$$Q(T,Z)=\eta\operatorname{sech}\eta[T+\kappa Z+\varepsilon\beta_1(\eta^2-3\kappa^2)Z-\theta_0]$$
$$\times\exp\left[-i\kappa T+\frac{i}{2}(\eta^2-\kappa^2)Z+i\varepsilon\beta_1\kappa(\kappa^2-3\eta^2)Z-i\sigma_0\right] \qquad (3.21)$$

With the above observation, one may expect that the PNLS equation (1.1) with $\Gamma=0$ can be approximated by a completely integrable system (3.20) up to order ε.[11] In fact, there is a map Φ transforming (3.20) into (1.1) with $\Gamma=0$, such that

$$q=\Phi(Q)=Q-\varepsilon i\left(3\beta_1-\frac{1}{2}\beta_2+\frac{1}{2}\beta_3\right)\frac{\partial Q}{\partial T}$$
$$-\varepsilon i(6\beta_1-\beta_2)Q\int_{-\infty}^{T}|Q|^2\,dT'+O(\varepsilon^2) \qquad (3.22)$$

Thus, up to order ε, the solution of (1.1) with $\Gamma=0$ can be expressed in terms of that of (3.20) which can be solved exactly (i.e., the PNLS equation without the loss may be said to be integrable up to order ε). It can be also shown that the map (3.22) is a canonical transformation with an appropriate choice of Hamiltonian structure for the PNLS equation. This situation is similar to that of finite-dimensional Hamiltonian system with perturbations consisting of homogeneous polynomials (theory of the Birkhoff normal form expansion). (See Ref. 12 for more discussion on this subject.)

4. RESHAPING OF THE SOLITONS

Here we show that the solitons deformed by the fiber loss can be reshaped by periodic excitations in the course of transmission through a fiber. From the discussion in the previous section, we consider the following set of equations for a transmission system with periodic excitations:

$$i\frac{\partial Q}{\partial Z}+\frac{1}{2}\frac{\partial^2 Q}{\partial T^2}+|Q|^2 Q$$
$$= \varepsilon i\beta_1\left(\frac{\partial^3 Q}{\partial T^3}+6\,|Q|^2\frac{\partial Q}{\partial T}\right)-i\Gamma Q, \qquad \text{for} \quad Z_l<Z<Z_{l+1} \tag{4.1}$$

and the relation of the excitations,

$$Q(T, Z_l+0)=Q(T, Z_l-0)+G(T, Z_l-0) \tag{4.2}$$

where $Z_l = l\Delta Z$ ($l=0, 1, 2,..., N$) are the positions of the devices for excitations described by the functions $G_l(T)=G(T, Z_l-0)$, and ΔZ the device spacing. The effect of the excitation $G_l(T)$ to the soliton (i.e., the change of the soliton parameters) can be analyzed by use of the eigenvalue problem in the IST method. [Recall that (4.1) is integrable if $\Gamma=0$.] The soliton parameters $\eta_l=\eta(Z_l-0)$ and $\kappa_l=\kappa(Z_l-0)$ in (3.21) for $Q_l(T)=Q(T, Z_l-0)$ are given by the eigenvalue $\zeta_l=\zeta(Z_l-0)=(\kappa_l+i\eta_l)/2$ in the eigenvalue problem,[9]

$$\begin{aligned} i\frac{\partial}{\partial T}\Psi_1-iQ_l\Psi_2&=\zeta_l\Psi_1\\ -i\frac{\partial}{\partial T}\Psi_2-iQ_l^*\Psi_1&=\zeta_l\Psi_2 \end{aligned} \tag{4.3}$$

We compute the new eigenvalue $\zeta(Z_l+0)$ from (4.3) with $Q(T, Z^l+0)$ given by (4.2) in the sense of perturbation for small $G_l(T)$ (the order of G_l is the same order as the loss rate of the amplitude). Then the variation $\Delta\zeta_l=\zeta(Z_l+0)-\zeta(Z_l-0)$ can be calculated by the formula obtained from (4.3),

$$\Delta\zeta_l=\int_{-\infty}^{\infty}(G_l\Psi_{2l}^2+G_l^*\Psi_{1l}^2)\,dT\bigg/\left(2i\int_{-\infty}^{\infty}\Psi_{1l}\Psi_{2l}\,dT\right) \tag{4.4}$$

where the eigenfunctions $\Psi_{il}=\Psi_i(T, Z_l-0)$ ($i=1, 2$) are those of the bound state solution of (4.3). If $Q_l(T)$ is a one-soliton solution given in the form

$$Q_l(T)=\eta_l\,\text{sech}\,\eta_l(T-\theta_l)\exp(i\sigma_l) \tag{4.5}$$

then the eigenfunctions Ψ_{il} are given by

$$\begin{aligned}\Psi_{1l}(T) &= -\left(\frac{\eta_l}{2}\right)^{1/2} \operatorname{sech} \eta_l(T-\theta_l) \exp\left(-\frac{1}{2}\eta_l T + \frac{i}{2}\sigma_l\right)\\ \Psi_{2l}(T) &= \left(\frac{\eta_l}{2}\right)^{1/2} \operatorname{sech} \eta_l(T-\theta_l) \exp\left(\frac{1}{2}\eta_l T - \frac{i}{2}\sigma_l\right)\end{aligned} \tag{4.6}$$

[Note that (4.5) agrees with the form (3.21) with $\kappa=0$.] From (4.1) and (4.2), one can define an infinite-dimensional map for the function $Q_l(T)$, $l=0, 1, 2,\ldots$, that is,

$$Q_{l+1}(T) = F_{\Delta Z}[Q_l(T) + G_l(T)] \tag{4.7}$$

where $F_Z[\cdot]$ is the solution of the initial value problem for (4.1), i.e., $Q(T, Z) = F_Z[Q(T, 0)]$ (the one-parameter group of diffeomorphism). The complete analysis of this map is practically impossible, because of its infinite-dimensional aspect [even (4.1) can be solved by the IST method]. However, if the functions $Q_l(T)$ are assumed to be the solitons having the form (4.5), then the map (4.7) can be reduced into a finite-dimensional map for the soliton parameters $\{\eta_l, \theta_l, \sigma_l\}$ in (4.5). [This assumption may not be valid for long distance, but it can be a good approximation for the field $Q_l(T)$ in the distances $Z \lesssim O(\Gamma^{-1})$ where the soliton part dominates the radiations produced by the loss and the excitations.[4]] From (3.12), (3.17), and (4.4), the finite-dimensional map can be expressed by

$$\begin{aligned}\eta_{l+1} &= \eta(Z_l+0)\, e^{-2\Gamma\Delta Z} = (\eta_l + \Delta\eta_l)\, e^{-2\Gamma\Delta Z}\\ \theta_{l+1} &= \theta_l - \varepsilon\beta_1 \frac{\eta_{l+1}^2}{4\Gamma}(e^{4\Gamma\Delta Z}-1)\\ \sigma_{l+1} &= \sigma_l + \frac{\eta_{l+1}^2}{8\Gamma}(e^{4\Gamma\Delta Z}-1) \qquad (\text{mod } 2\pi)\end{aligned} \tag{4.8}$$

where $\Delta\eta_l$ is the imaginary part of $2\Delta\zeta_l$ in (4.4), and a function of θ_l and σ_l. Here we have assumed that $\Delta\zeta_l$ is pure imaginary, i.e., $\Delta\zeta_l = i\Delta\eta_l/2$, to be consistent with the assumption (4.5) [this can be achieved by taking $G_l(T)$ to be symmetric in $T-\theta_l$]. In the remaining part of this section, we give two examples of such finite-dimensional maps corresponding to the transmission systems with periodic excitations.

As the first example, we consider an amplification with nonlinear suppression given by the following form of G_l:

$$G_l(T) = \alpha(1 - \delta\,|Q_l(T)|^2)\, Q_l(T) \tag{4.9}$$

where α and δ ($\ll 1$) are linear and nonlinear gains, and both positive real constants. From (4.4) and (4.9), we obtain

$$\Delta\eta_l = 2\alpha\left(1 - \frac{2}{3}\delta\eta_l^2\right)\eta_l \tag{4.10}$$

Consequently in this case, we have a one-dimensional map for η_l given by

$$\eta_{l+1} = e^{-2\Gamma\Delta Z}\left[1 + 2\alpha\left(1 - \frac{2}{3}\delta\eta_l^2\right)\right]\eta_l \tag{4.11}$$

This kind of map (e.g., quadratic map) has been studied extensively as a model of turbulence (chaos), and shown to have the period-doubling bifurcations corresponding to various values of control parameters α and δ. In the case of (4.11), if α satisfies an inequality, $\alpha > [\exp(2\Gamma\Delta Z) - 1]/2 \sim \Gamma\Delta Z$, then there is a fixed point η_* such that

$$\eta_*^2 = \frac{3}{2\delta}\left[1 - \frac{\exp(2\Gamma\Delta Z)}{1 + 2\alpha}\right] \sim \frac{3(\alpha - \Gamma\Delta Z)}{\delta(1 + 2\alpha)} \tag{4.12}$$

One can also show that the fixed point (4.12) is stable for certain regions of α and δ, e.g., as a practical case, $\alpha = [\exp(2\Gamma\Delta Z) - 1 + 2\delta/3]/2$ and sufficiently small δ (note that the choice of α gives $\eta_* = 1$). Thus the solitons can be reshaped into a unique soliton whose amplitude is η_*. It should be noted that this unification of solitons is necessary to keep each soliton separated, since the higher-order terms in (1.1) produce the relative velocity between two solitons with different amplitudes [see (3.21)].

The second example is the case in which the excitations are given externally by the function

$$G_l(T) = \alpha g(T)\, e^{i\beta} \tag{4.13}$$

where $g(T)$ is a real symmetric function with $\max|g(T)| = 1$, and $\alpha(\ll 1)$, β are positive real constants. For a simple case, we consider (4.1) with $\varepsilon = 0$. In this case, from (4.4) and (4.11), we have a two-dimensional map for $\{\eta_l, \sigma_l\}$ (where $\theta_{l+1} = \theta_l = \text{const}$ is chosen to be zero)

$$\eta_{l+1} = \eta_l\left[1 + \alpha\cos(\beta - \sigma_l)\int_{-\infty}^{\infty} g(T)\,\mathrm{sech}\,\eta_l T\, dT\right] e^{-2\Gamma\Delta Z} \tag{4.14}$$

and σ_l is given in (4.8). This map is similar to the one given in Ref. 13, and depending on the control parameters α and β, it has many interesting properties, such as, fixed point, regular, and strange attractors. (See also the paper given by McLaughlin in this proceedings.)

These models of the periodic excitations have been studied numerically in the case of (4.1) with $\varepsilon = 0$.[3-5] The results showed that the shape of the soliton remains amazingly persistent even for the distance $Z \simeq N\Delta Z \simeq O(\Gamma^{-2})$ [the distance considered in this paper is of $O(\Gamma^{-1})$] (N is the number of amplifications). In this scale of distance, the radiation which is considered as an order Γ^2 plays an important role to maintain the structure of the soliton,[4] that is, one can*not* reduce the infinite-dimensional map (4.7) to the map consisting of the soliton parameter only (4.8). (An analytical study of the radiation effects to the soliton at the second order still needs to be carried out.)

As a final remark, recently Hasegawa proposed another amplification process by use of the stimulated Raman process where the fiber loss can be suppressed by the pump field injected externally[14]. He showed that the Raman process gives an adiabatic process of amplification, and the spacing of the amplifiers depending on the loss rate can be taken much longer than the one in the example of the discrete amplification discussed here.

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